

Geometric characterization of separability and entanglement in pure Gaussian states by single-mode unitary operations

Gerardo Adesso,^{1,2,3} Salvatore M. Giampaolo,^{4,2} and Fabrizio Illuminati^{4,2,5,*}

¹*Dipartimento di Fisica “E. R. Caianiello”, Università degli Studi di Salerno, Via S. Allende, I-84081 Baronissi (SA), Italy*

²*CNR-INFM Coherentia, Napoli, Italy; CNISM, Unità di Salerno; and INFN, Sezione di Napoli - Gruppo Collegato di Salerno, Italy*

³*Grup d’Informació Quàntica, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain*

⁴*Dipartimento di Matematica e Informatica, Università degli Studi di Salerno, Via Ponte don Melillo, I-84084 Fisciano (SA), Italy*

⁵*ISI Foundation for Scientific Interchange, Viale Settimio Severo 65, I-10133 Turin, Italy*

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We present a geometric approach to the characterization of separability and entanglement in pure Gaussian states of an arbitrary number of modes. The analysis is performed adapting to continuous variables a formalism based on single subsystem unitary transformations that has been recently introduced to characterize separability and entanglement in pure states of qubits and qutrits [arXiv:0706.1561]. In analogy with the finite-dimensional case, we demonstrate that the $1 \times M$ bipartite entanglement of a multimode pure Gaussian state can be quantified by the minimum squared Euclidean distance between the state itself and the set of states obtained by transforming it via suitable local symplectic (unitary) operations. This minimum distance, corresponding to a, uniquely determined, extremal local operation, defines a novel entanglement monotone equivalent to the entropy of entanglement, and amenable to direct experimental measurement with linear optical schemes.

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I. INTRODUCTION

The concept of entanglement has gradually developed from the status of a puzzling interpretational problem, to that of a crucial operational resource for quantum information tasks and, even more remarkably, to the status of a founding property of quantum theory, whose implications and applications extend into many diverse areas of research ranging from quantum optics and atomic and molecular physics to condensed matter physics and quantum critical phenomena [1, 2]. While many open questions, even on defining grounds, stand open when it comes to address questions like the nature of multipartite entanglement and the entanglement of mixed states, a fairly satisfactory classification and quantification of bipartite entanglement of pure quantum states have been established [2, 3]. This achievement has been partly possible because the milestones of quantum information science, like quantum teleportation, quantum cryptography, state transfer, broadcasting and telecloning, entanglement creation and distillation, all rely on the paradigm of two distant labs operated by two parties – traditionally named Alice and Bob – who wish to communicate, possibly sharing a pure entangled state [4]. By properly defining figures of merit associated with such protocols, pure-state bipartite entanglement has been understood both qualitatively – entangled states are non-separable – and quantitatively – the degree of pure-state bipartite

entanglement is equal to the entropy of the reduced state of each subsystem. In particular, the von Neumann entropy of entanglement is equal both to the, operationally defined, distillable entanglement and entanglement cost of pure bipartite quantum states [5]. This equivalence is lost in the presence of mixedness, and the phenomenon of entanglement conversion irreversibility sets in [6].

There exists an alternative, equally natural way to understand and characterize entanglement. From a *geometric* perspective, the degree of entanglement in a state ρ can be quantified as the minimum distance, suitably measured, between ρ and the set of unentangled, separable states [3]. Again, for pure states, if such distance is measured in terms of the relative entropy, the resulting measure of entanglement coincides with the von Neumann entropy of entanglement [7]. This suggests that other entanglement monotones, that can be useful either for their operational meaning and/or for their computability, might be endowed with an alternative, geometric interpretation which adds to their understanding and may provide alternative tools in their experimental detection. A novel approach to the study of this problem has been recently introduced for low-dimensional discrete-variable systems such as qubits and qutrits [8]. It relies on the basic idea that entanglement can be characterized by the response of a system to *local* and *unitary* perturbations that, by definition, cannot change the degree of entanglement present in the system. Notwithstanding this simple fact, oddly enough, degrees of freedoms that are affected by local unitaries tend to be systematically neglected in the analyses of entanglement properties [9]. The recent study by Giampaolo and Illuminati [8] yields instead that

*Corresponding author. Electronic address: illuminati@sa.infn.it

there exist specific single-party unitary operations (corresponding to the projection on the \hat{z} component of the spin for qubits and qutrits) which have the following properties: (i) they leave a pure bipartite state invariant if and only if it is a product state; (ii) they transform any pure bipartite entangled state in such a way that the minimum squared Euclidean distance of the original state from the set of all possible transformed states is an entanglement monotone. In the case of bipartite states of qubits and qutrits, such a measure coincides exactly with the marginal linear entropy, quantifying the degree of impurity (mixedness) of each subsystem [8]. Therefore, entanglement monotones based on completely different definitions, such as the linear entropy and the tangle [10], are re-discovered and re-interpreted in terms of the Hilbert-space distance between quantum states and their images under suitably selected local unitary operations.

In this work, we apply the framework introduced in Ref. [8] to characterize entanglement of pure Gaussian states of continuous-variable systems. Recent progresses have showed that many nontrivial problems in entanglement theory, whose remarkable complexity renders their solution unachievable in qudit systems with d greater than 2 or 3, can be successfully tackled with different techniques when considering systems defined on infinite-dimensional Hilbert spaces, like, e.g. the quantum electromagnetic field [11]. In particular, Gaussian states, such as coherent, squeezed states, and in general all ground and thermal states of harmonic lattices, have played an increasingly important role in quantum information science, thanks to their simple structural properties as well as to the high degree of experimental control on their production and manipulation [12, 13]. Motivated by these considerations, we seek here to provide a novel geometric interpretation for bipartite entanglement of pure Gaussian states, in terms of the perturbation induced on them by single-mode unitary operations in Hilbert space, or, equivalently, symplectic transformations in quantum phase space. We will find, in direct analogy with the discrete-variable case [8], that there exists a single-mode symplectic operation which preserves product states, and whose action leads in general to the definition of a pure-state entanglement monotone for $1 \times N$ Gaussian states. This measure does not exactly coincide with any known entanglement measure, even though it is a monotonically increasing function of the entropy of entanglement, providing thus a novel quantifier of continuous variable entanglement endowed with a purely geometric interpretation.

The paper is organized as follows: in Section II we briefly review the basic tools of the symplectic formalism in phase space, that is best suited for the analysis of separability, entanglement, and quantum operations on Gaussian states of infinite-dimensional quantum systems. In section III we introduce and analyze the properties of single-mode unitary (symplectic) operations in quantum phase space, and define the distance, induced by the fidelity, between pure Gaussian states and their images

under such operations (these images are again pure Gaussian states). We then proceed to determine the minimum distance over the set of all possible such transformations, and the associated *extremal* operation. We prove that invariance of a state under the action of the extremal operation is a necessary and sufficient condition for the full separability of multimode pure Gaussian states of translationally invariant system, and show that the associated minimum distance is an entanglement monotone closely related to the linear entropy of the subsystem reductions. We finally discuss the relation between this novel entanglement monotone and the various possible extensions of the definition of the tangle to continuous variable systems. In Section IV we point out at some possible future lines of investigation in the framework of the formalism of local symplectic operations, also concerning mixed states, and discuss possible methods for the direct experimental detection of the minimum distance using linear optical elements, with an explicit example focused on tripartite Gaussian states.

II. PHASE-SPACE DESCRIPTION OF GAUSSIAN STATES AND SINGLE-MODE SYMPLECTIC OPERATIONS

We consider a continuous-variable (CV) system consisting of N canonical bosonic modes, associated with an infinite-dimensional Hilbert space, tensor product of the N single-mode Fock spaces [11, 12, 14]. Unitary operations which are at most quadratic in the canonical operators, amount to symplectic transformations in phase space. A real $2N \times 2N$ matrix describes a symplectic transformation $S \in Sp(2N, \mathbb{R})$ if, by definition, it preserves the symplectic form,

$$S\Omega S^T = \Omega, \quad \Omega = \omega^{\oplus N}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1)$$

For a single mode, the generators of the symplectic group $Sp(2, \mathbb{R})$ are [15]

$$\Sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2)$$

where $\Sigma_2 = \omega$. The matrices Σ_i 's in Eq. (2) are traceless. Together with the identity matrix \mathbb{I} , they form a basis in the space of 2×2 real matrices. According to the Euler decomposition, the most general single-mode symplectic operation $S \in Sp(2, \mathbb{R})$ can be written as a sequence of a rotation, a squeezing, and a second rotation (with different angle) in phase space,

$$S = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (3)$$

reducing to the identity transformation for $\theta = \phi = 0$, $\xi = 1$.

We are interested in studying the minimal distance between a state and its image as transformed by a specific

type of local single-mode symplectic operations. Clearly, one cannot allow the identity transformation in the defining set of possible operations, if one wants to avoid ending up with a trivial null distance on all quantum states. Then, in analogy with the finite-dimensional case, we impose the condition of tracelessness [8], and we define a *unitary single-mode operation* S_{smo} as the most general $Sp(2, \mathbb{R})$ symplectic transformation of the form Eq. (3), with $\text{Tr } S = 0$. In this way we are only considering symplectic transformations which are orthogonal to the identity. Imposing such constraint yields $\phi = \pi/2 - \theta$, namely

$$S_{smo} = \begin{pmatrix} \frac{(\xi^2 - 1) \cos \theta \sin \theta}{\xi} & \frac{\cos^2 \theta + \xi^2 \sin^2 \theta}{\xi} \\ -\frac{\xi^2 \cos^2 \theta + \sin^2 \theta}{\xi} & -\frac{(\xi^2 - 1) \xi \cos \theta \sin \theta}{\xi} \end{pmatrix}. \quad (4)$$

The transformation $S_{smo}(\xi, \theta)$ can be written as a linear combination of the Σ_i 's from Eq. (2), $S_{smo} = \alpha \Sigma_1 + \beta \Sigma_2 + \gamma \Sigma_3$, where the symplectic condition Eq. (1) imposes $\gamma = \sqrt{\beta^2 - \alpha^2 - 1}$, $\beta \geq \sqrt{\alpha^2 + 1}$. Explicitly [36]:

$$S_{smo}(\alpha, \beta) = \begin{pmatrix} \sqrt{\beta^2 - \alpha^2 - 1} & \alpha + \beta \\ \alpha - \beta & -\sqrt{\beta^2 - \alpha^2 - 1} \end{pmatrix}, \quad (5)$$

where the parameters α, β are connected with the squeezing ξ and the rotation angle θ , appearing in Eq. (4), by the following relations:

$$\begin{aligned} \xi &= [\beta\alpha + \alpha\sqrt{\beta^2 - 1}]/\alpha, \\ \cos \theta &= \sqrt{[\beta^2 - \alpha\sqrt{\beta^2 - 1} - 1]/[2(\beta^2 - 1)]}. \end{aligned} \quad (6)$$

III. EXTREMAL SINGLE-MODE OPERATIONS AND ENTANGLEMENT OF PURE GAUSSIAN STATES

We can now move to the specific setting of the geometric analysis. Let our N -mode bosonic system be prepared in a pure Gaussian state [11]. We recall that Gaussian states of N modes are completely described in phase space (once the first moments are set to zero via local displacements) by the real, symmetric $2N \times 2N$ covariance matrix (CM) σ , whose entries are $\sigma_{lm} = 1/2 \langle \{\hat{X}_l, \hat{X}_m\} \rangle - \langle \hat{X}_l \rangle \langle \hat{X}_m \rangle$. Here $\hat{X} = \{\hat{x}_1, \hat{p}_1, \dots, \hat{x}_N, \hat{p}_N\}$ is the vector of the field quadrature operators, whose canonical commutation relations can be expressed in matrix form: $[\hat{X}_l, \hat{X}_m] = 2i\Omega_{lm}$, with the symplectic form Ω defined in Eq. (1). According to Williamson theorem [16], the CM of a N -mode Gaussian state can be always diagonalized by means of a global symplectic transformation (this corresponds to the normal mode decomposition): $W_\sigma \sigma W_\sigma^T = \nu$, where $W_\sigma \in Sp(2N, \mathbb{R})$ and $\nu = \bigoplus_{k=1}^N \text{diag}\{\nu_k, \nu_k\}$ is the CM corresponding to the tensor product of single-mode thermal states. The quantities $\{\nu_k\}$ are the so-called symplectic eigenvalues of the CM σ .

A *pure* Gaussian state is characterized by $\nu_k = 1$, $\forall k = 1 \dots N$, which implies $\text{Det } \sigma = 1$. Such a state may be, for instance, the ground state of some harmonic Hamiltonian. We want to study the $1 \times (N-1)$ entanglement of one mode with the remaining $N-1$ modes, via the perturbation induced by single-mode operations on mode 1. Namely, we aim to study the minimal squared distance between the Gaussian state σ and the state obtained from it by applying a S_{smo} of the form Eq. (5) on any selected mode, for instance mode 1. It is important to recall that the transformed state, being obtained from the original pure Gaussian state by applying to it a symplectic transformation, i.e. a unitary transformation at most quadratic in the field variables, is again a pure Gaussian state. Introducing the standard Bures metric, the minimum distance reads

$$D(\sigma) = \min_{\alpha, \beta} [1 - \mathcal{F}(\sigma, \sigma')] \quad (7)$$

Here $\sigma' = [S_{smo}(\alpha, \beta) \oplus \mathbb{1}_{2 \dots N}] \cdot \sigma \cdot [S_{smo}(\alpha, \beta) \oplus \mathbb{1}_{2 \dots N}]^T$, and the *fidelity* \mathcal{F} between two pure-state N -mode Gaussian CMs can be computed as [17]

$$\mathcal{F}(\sigma, \sigma') = 2^N / \sqrt{\text{Det}(\sigma + \sigma')},$$

amounting to the overlap $|\langle \psi | \psi' \rangle|^2$ between the original and the perturbed Gaussian quantum states.

To proceed in the evaluation of Eq. (7), let us first take into account that, in full generality, pure Gaussian states can always be brought in the phase-space Schmidt form [11] with respect to the $1 \times (N-1)$ bipartition. The symplectic transformation W achieving the Schmidt decomposition is the direct sum of the two Williamson diagonalizing operations acting on the single-mode and the $(N-1)$ -mode subspaces, respectively, $W = W_1 \oplus W_{2 \dots N}$. One then has

$$\begin{aligned} \sigma_W &= W \sigma W^T \\ &= \begin{pmatrix} a & 0 & \sqrt{a^2 - 1} & 0 \\ 0 & a & 0 & -\sqrt{a^2 - 1} \\ \sqrt{a^2 - 1} & 0 & a & 0 \\ 0 & -\sqrt{a^2 - 1} & 0 & a \end{pmatrix} \\ &\oplus \mathbb{1}_{3 \dots N}, \quad \text{with } a \geq 1, \end{aligned} \quad (8)$$

i.e. the phase-space Schmidt form of σ is constituted by one two-mode squeezed state between modes 1 and 2, tensor $N-2$ uncorrelated vacua [18]. To evaluate Eq. (7), we need the minimum of $\text{Det}[\sigma + (S_{smo} \oplus \mathbb{1}_{2 \dots N}) \sigma (S_{smo} \oplus \mathbb{1}_{2 \dots N})^T]$. We will now show that it is enough to consider states in the form σ_W . In fact,

$$\begin{aligned}
& \text{Det} [\sigma_W + (S_{smo} \oplus \mathbb{1}_{2\dots N}) \sigma_W (S_{smo} \oplus \mathbb{1}_{2\dots N})^T] \\
&= \text{Det} [(W_1 \oplus W_{2\dots N}) \sigma (W_1^T \oplus W_{2\dots N}^T) + (S_{smo} W_1 \oplus W_{2\dots N}) \sigma (W_1^T S_{smo}^T \oplus W_{2\dots N}^T)] \\
&= \text{Det} [(W_1 \oplus W_{2\dots N}) \sigma (W_1^T \oplus W_{2\dots N}^T) + (W_1 W_1^{-1} S_{smo} W_1 \oplus W_{2\dots N}) \sigma (W_1^T S_{smo}^T W_1^{-1T} \oplus W_{2\dots N}^T)] \\
&= \text{Det} \{ [W_1 \oplus W_{2\dots N}] [\sigma + (W_1^{-1} S_{smo} W_1 \mathbb{1}_{2\dots N}) \sigma (W_1^{-1} S_{smo} W_1 \oplus \mathbb{1}_{2\dots N})^T] [W_1 \oplus W_{2\dots N}]^T \} \\
&= \text{Det} [\sigma + (W_1^{-1} S_{smo} W_1 \oplus \mathbb{1}_{2\dots N}) \sigma (W_1^{-1} S_{smo} W_1 \oplus \mathbb{1}_{2\dots N})^T], \tag{9}
\end{aligned}$$

where we exploited the group properties of $Sp(2, \mathbb{R})$, the fact that a symplectic operation S has $\text{Det } S = 1$, and the property that the inverse S^{-1} of a symplectic transformation S is itself symplectic. Now, from the cyclic property of the trace, it follows that $W_1^{-1} S_{smo} W_1$ is itself a traceless symplectic operation, i.e. a single-mode operation of the form Eq. (5). Thus the minimum of the above determinant, taken over the entire set $S_{smo}(\alpha, \beta)$ of single-mode unitary operations, is invariant under local symplectic operations $W_1 \oplus W_{2\dots N}$ performed on state σ . Thus, without loss of generality, we can choose a pure N -mode Gaussian state in the phase-space Schmidt form σ_W of Eq. (8). Therefore,

$$\begin{aligned}
& \text{Det} [\sigma_W + (S_{smo} \oplus \mathbb{1}_{2\dots N}) \sigma_W (S_{smo} \oplus \mathbb{1}_{2\dots N})^T] \\
&= 2^{2(N-1)} [(a^2 - 1)^2 + 4a^2 \beta^2]. \tag{10}
\end{aligned}$$

The minimum is then acquired, as $\beta \geq \sqrt{\alpha^2 + 1}$, for $\beta = 1$, $\alpha = 0$. The corresponding extremal single-mode operation is then, finally

$$S_{smo}(0, 1) \equiv \Sigma_2 \equiv \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11}$$

This is a simple rotation of $\pi/2$ in phase space, and may be seen as the CV analogue of the spin-flip operation on qubits, realized by the σ_y Pauli matrix.

We observe that a product state, characterized by a CM in direct sum form, $\sigma^\oplus \equiv \sigma_1 \oplus \sigma_{2\dots N}$, is left *invariant* by the extremal single-mode operation:

$$[S_{smo}(0, 1) \oplus \mathbb{1}_{2\dots N}] \sigma^\oplus [S_{smo}(0, 1) \oplus \mathbb{1}_{2\dots N}]^T = \sigma^\oplus. \tag{12}$$

That is, on pure product Gaussian states, extremal and invariant (or preserving) operations coincide, in full analogy with the finite-dimensional case analyzed in Ref. [8]. Hence, *a pure Gaussian states is separable if and only if there exists a traceless single-mode symplectic (unitary) operation that leaves it unperturbed*. This is again in perfect analogy with the discrete-variable analysis performed for qubits and qutrits [8]. The minimum distance Eq. (7), achieved for $\beta = 1$, $\alpha = 0$, can now be evaluated explicitly and reads

$$D(\sigma) = 1 - \frac{2^N}{2^{N-1}(a^2 + 1)} = \frac{a^2 - 1}{a^2 + 1}. \tag{13}$$

The quantity $D(\sigma)$ is a *measure* of the entanglement between mode 1 and the rest of the system, being an

increasing function of the single-mode mixedness factor $a \geq 1$. For product states $a = 1$ and one correctly retrieve $D(\sigma^\oplus) = 0$. One should recall that Eq. (13) holds in general, and not only for states in Schmidt form, once a is identified with the (locally invariant) symplectic eigenvalue of the reduced CM σ_1 of mode 1, computable from its determinant as $a = \sqrt{\text{Det } \sigma_1}$. From this observation, it follows immediately that $D(\sigma)$, even if constructed through the action of local unitaries, is invariant under them – as already proved by Eq. (9) – and is thus a proper entanglement measure.

The $1 \times (N - 1)$ linear entropy of entanglement for the state σ (corresponding to the tangle for qubits [10]) reads

$$E_L(\sigma) = \frac{a - 1}{a}. \tag{14}$$

We see that E_L is a monotonic function of D , thus qualitatively equivalent to but yet not exactly coincident with the latter everywhere, at variance with the discrete-variable case, in which they do strictly coincide [8] (The behavior of the two entanglement monotones is compared in Fig. 1). The fact that the linear entropy of the reduced state does not coincide exactly with the minimum distance achieved under local symplectic operations may be traced back to the non uniqueness in the definition of the “tangle” for Gaussian states of CV systems. For

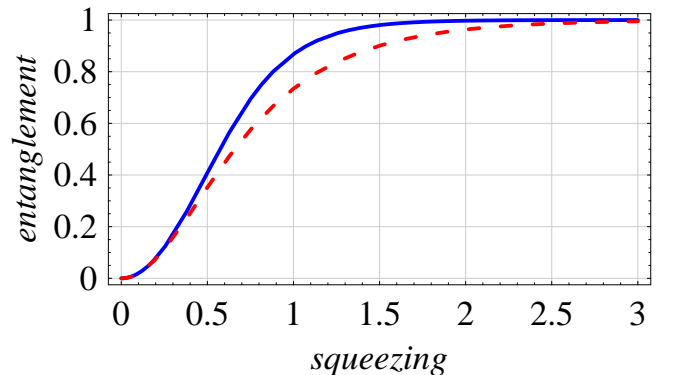


FIG. 1: (color online) Entanglement of pure $(1 \times (N - 1))$ -mode Gaussian states as a function of the single-mode squeezing r , defined such that $a = \cosh 2r$. The solid line depicts the distance-based geometric measure D defined in Eq. (13), while the dashed line corresponds to the linear entropy of entanglement E_L , Eq. (14).

qubits, at least four different definitions coalesce into the same entanglement monotone: (i) squared concurrence [10]; (ii) local linear entropy [19]; (iii) squared negativity (negativity equals concurrence for pure qubit states [20]); (iv) minimum distance under single-qubit unitary transformation [8]. On the other hand, while the concurrence is not well defined in CV systems, the other definitions of the tangle all give rise to different (yet equivalent) entanglement quantifiers in these systems. For instance, the Gaussian tangle defined as the squared negativity [21], in analogy with definition (iii), reads

$$\tau_G(\sigma) = [a^2 + a\sqrt{a^2 - 1} - 1]/2. \quad (15)$$

The von Neumann entropy of entanglement, for reference, is given by $E_V(\sigma) = [(a+1)/2] \log[(a+1)/2] - [(a-1)/2] \log[(a-1)/2]$. All these measures are monotonically increasing functions of each other (and of a), some of them being normalized between 0 and 1 (like D and E_L), the others diverging in the limit of infinite squeezing, $a \rightarrow \infty$.

IV. EXPERIMENTAL REMARKS AND FUTURE PERSPECTIVES

The minimum distance D provides a new *bona fide* geometric measure of entanglement for pure Gaussian states, close in spirit to the low-dimensional, discrete-variable counterpart introduced in Ref. [8], and embodying yet another generalization of the tangle. However, we would like to remark that, among the three possible CV versions of the tangle, only τ_G , Eq. (15), satisfies the CV generalization of the Coffman-Kundu-Wootters monogamy inequality [10, 19], as proved in Ref. [21] for all, pure and mixed, N -mode Gaussian states.

On the other hand, the geometric measure of entanglement $D(\sigma)$ that we have introduced in this work for pure Gaussian states, has the nice property of being amenable to direct experimental investigation, once two copies of an unknown Gaussian state with CM σ are available. One first needs a phase plate in order to rotate one copy of $\pi/2$, realizing the operation Σ_2 , as demonstrated e.g. in [22]. Thereafter, the evaluation of the overlap between the rotated copy and the unrotated one involves standard tools of linear optics, as routinely demonstrated in the determination of the fidelity \mathcal{F} of teleportation experiments with continuous variables [23], or in the implementation of interferometric schemes [24] that can be realized even without homodyning [25]. Our result thus suggests a way to the direct measurement of CV entanglement in pure $1 \times N$ Gaussian states, in analogy with what achieved experimentally in the case of qubits: In that case, the entanglement, quantified by the two-point concurrence, has been directly measured on the two-fold copy of unknown two-qubit pure states [26].

This proposal looks especially appealing for Gaussian states with a small number of modes. A relevant example is provided by three-mode Gaussian states, whose CM

assumes in general the following expression in terms of 2 by 2 submatrices,

$$\sigma = \begin{pmatrix} \sigma_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12}^\top & \sigma_2 & \epsilon_{23} \\ \epsilon_{13}^\top & \epsilon_{23}^\top & \sigma_3 \end{pmatrix}. \quad (16)$$

The structural and informational properties of three-mode Gaussian states, with a special emphasis on the pure-state instance, have been extensively studied in Ref. [27], while a scheme for their production via interlinked nonlinear interactions in $\chi^{(2)}$ media has been presented in Ref. [28], together with preliminary experimental results. When modes 2 and 3 have the same average number \bar{n} of thermal photons, the corresponding (parametric) pure three-mode Gaussian state is said to be ‘bisymmetric’ and its CM can be written in the standard form of Eq. (16), with

$$\begin{aligned} \sigma_1 &= a \mathbb{1}_2, \\ \sigma_2 = \sigma_3 &= \left(\frac{a+1}{2} \right) \mathbb{1}_2, \\ \epsilon_{23} &= \left(\frac{a-1}{2} \right) \mathbb{1}_2, \\ \epsilon_{12} = \epsilon_{13} &= \text{diag} \left\{ \frac{\sqrt{a^2-1}}{\sqrt{2}}, -\frac{\sqrt{a^2-1}}{\sqrt{2}} \right\}, \end{aligned}$$

$$\text{and } a = 4\bar{n} + 1.$$

The geometric entanglement between the first mode and the group of modes 2 and 3, as obtainable from the single-mode perturbation Eq. (11) applied on mode 1, is then directly given by Eq. (13) as a function of \bar{n} . The three-mode Gaussian states of this family are known to be optimal resources for $1 \rightarrow 2$ CV telecloning (i.e. cloning at distance, or equivalently teleportation to more than one receiver) of single-mode coherent states [29], as discussed also in [27, 28]. The single-clone fidelity \mathcal{F} exhibits a non-monotonic, concave behaviour as a function of \bar{n} , reaching the maximum $\mathcal{F}^{\max} = 2/3$ at $\bar{n} = 1/2$. Very recently, the first experimental demonstration of unconditional $1 \rightarrow 2$ telecloning of unknown coherent states has been realized [30], with a measured fidelity for each clone of $\mathcal{F} = 0.58 \pm 0.01$ (surpassing the classical threshold of 0.5). This experimental milestone has raised renewed interest towards CV quantum communication [31]. In the context of this work, such an achievement entails that all the elementary steps required to access pure-state Gaussian entanglement from a geometric point of view have been already successfully undertaken. Our prescription, therefore, is likely to be seen “at work” experimentally on multimode Gaussian states in the near future.

In this paper we have dealt with *pure* Gaussian states only. It is natural to ask whether a suitable generalization of the present approach is able to provide a geometric interpretation, possibly amenable to direct experimental tests, of *mixed*-state entanglement measures as well. In this respect, it is important to clarify to which extent the

results of this paper are still valid for mixed states. *In primis*, it is generally true that the extremal single-mode operation, Eq. (11), preserves tensor product, even mixed Gaussian states [see Eq. (12)]. However, convex combination of product states, i.e. separable mixed Gaussian states, are not left invariant by the action of such local operation. Accordingly, the corresponding geometric measure (minimum distance D) defined by Eq. (13) is not, in general, an entanglement monotone for mixed Gaussian states. One can thus conclude that, in the mixed-state scenario, the mere action of $S_{smo}(0, 1)$ leads to a distinction between tensor product states (totally uncorrelated, on which the distance Eq. (13) is zero) and states displaying some form of (quantum and/or classical) correlation. A refinement is henceforth necessary in order to aim at discriminating, from a geometric point of view, the quantum portion – entanglement – from the total amount of correlations. A feasible way to deal with this issue seems that of identifying a minimal set of single-mode unitary operations, such that a suitably defined “distance” involving their combined action, may turn to be equivalent (or to provide bounds) to known entanglement monotones (e.g. negativities, tangles and/or measures based

on the Gaussian convex roof [32]). One should then be able to readily provide a recipe for the practical estimation of mixed-state Gaussian entanglement with few local measurements (see also [33]).

Finally, we would like to remark that the framework introduced in Ref. [8], and further discussed in the present paper, can be naturally applied to investigate criticality and entropy scaling in the ground states of harmonic lattices [34], with the purpose of establishing connections similar to those unveiled for the ground states of spin systems at criticality [35]. This interesting perspective will be the object of further future studies.

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